Binaries and core-ring structures in self-gravitating systems

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Low-energy states of self-gravitating systems with finite angular momentum are considered. A constraint is introduced to confine cores and other condensed objects within the system boundaries by gravity alone. This excludes previously observed astrophysically irrelevant asymmetric configurations with a single core. We show that, for an intermediate range of a short-distance cutoff and small angular momentum, the equilibrium configuration is an asymmetric binary. For larger angular momentum or for a smaller range of the short-distance cutoff, the equilibrium configuration consists of a central core and an equatorial ring. The mass of the ring varies between zero for vanishing rotation and the full system mass for the maximum angular momentum L_{max} a localized gravitationally bound system can have. The value of L_{max} scales as $\sqrt{\ln(1/x_0)}$, where x_0 is a ratio of a short-distance cutoff range to the system size. An example of the soft gravitational potential is considered; the conclusions are shown to be valid for other forms of short-distance regularization.

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I. INTRODUCTION

Despite an abundance of rotating structures in the universe, the statistical mechanics of these systems is not completely understood. The nonrotating self-gravitating systems, as shown by the mean-field (MF) analysis and recently confirmed by direct computer simulations, exhibit two phases (see, for example, Refs. [1–4] and references therein): A high-energy "uniform" phase where density contrast is small, and a low-energy phase consisting of a diluted halo and a dense core. The structure of the core is determined by a type of a short-range regularization, which can vary from the exclusion in phase or coordinate space to a small-distance truncation of the interaction potential. At the same time, the structure of rotating self-gravitating systems, particularly the form and number of cores or other dense objects in the low-energy states, remains a subject of discussion. Configurations such as symmetric and asymmetric binaries and rings [5–8], a single spheroidal core with or without an equatorial ring [10,11], and a single "asymmetric" core, sliding along the container wall [12], have been found in rotating self-gravitating systems. Among possible reasons for this plethora of equilibrium configurations, the boundary conditions definitely play an important role. As in the nonrotating case (see, for example, Ref. [1]), a confining box with reflecting walls (to conserve the energy and mass) is essential for the existence of any equilibrium state. For a rotating system, it is natural to require angular momentum conservation as well and choose an axisymmetric boundary; usually a spherical container is considered [5,11–13]. However, the reflective boundary conditions seem to be the main reason for the apparent discrepancy between the particle simulations and the MF results: While a true particle system inevitably evolves towards an asymmetric single-core configuration [12], the states with an arbitrary number of cores and even rings are predicted by the MF analysis [5–8,11]. To obtain two- and multicore states, the center of mass of the system is fixed in the center of the container [5–9]. While in the MF analysis the center of mass constraint is implemented by removing the dipole terms from the multipole expansions of density and potential, it is not clear how to fix the center of mass in a particle simulation. The total momentum and the center of mass position, unlike the angular momentum, are not conserved even in a spherical system, when the boundaries are reflecting. Due to the ergodicity of a three-dimensional Coulomb system, even a specially prepared, highly symmetric initial state (with centrally symmetric coordinates and momenta distribution) will evolve toward the usually nonsymmetric highest entropy configuration. For a rotating system, the most probable configuration consists of the single core sliding along the container wall [9,12]. The reason why the state with a single core has the highest entropy (or the lowest free energy) is the following: When two or more initially separated cores merge, the gravitational potential energy decreases, which, for the fixed total energy, leads to a gain in the translational entropy of the halo. Intuitively, as there is no naturally occurring analog of a container wall on which the core may slide, this asymmetric state with a single core looks highly unphysical.

If particle simulations seem unable to reproduce the configurations obtained by the MF methods and exhibit only a physically irrelevant state with a single core, is there any other way to validate the MF results? The criteria of physical relevance, which the asymmetric state with a single core does not satisfy, are rather intuitive and can be formulated as follows: The physically relevant equilibrium states of rotating systems must be affected by the boundary conditions in the least possible way. This minimal boundary condition effect is attained in the case of the core-halo states of nonrotating self-gravitating systems: If a container surrounding such system is removed, the halo will start to evaporate, while the core will remain almost intact for a considerable time (see Ref. [14] for an estimate of core-halo thermalization rate). On the contrary, the rotating asymmetric state with a single core would undergo significantly more dramatic changes if the container were removed: The core, no longer being supported by the container wall, will escape ballistically with nothing left within the system boundaries. Similarly to the nonrotating case, the minimal boundary condition effect can be attained in rotating systems if cores (or other condensed objects such as disks or rings) are confined within the system boundary by gravity alone. In the limit of the ground-state energy (or zero temperature), such a state will consist only of the gravitationally bound condensed parts, unaffected by the removal of the container at all.

In this paper we consider such weakly interacting with container, or "physically relevant" states of rotating self-gravitating systems. To satisfy the relevance criteria suggested above, one needs to find core orbits in the presence of a halo. To simplify this task, we consider the limit of ground-state energy (or zero temperature) in which the gaseous halo is condensed into the cores, and there is no internal motion of core particles. The only motion in this limit is the macroscopic movement of cores, specified by the angular momentum constraint. For sufficiently low energies, the state with the lowest ground-state energy is the thermodynamically equilibrium one (while the other mechanically stable states are thermodynamically metastable). Thus, to determine a core structure of the physically relevant low-energy equilibrium state, it is sufficient to find a gravitationally bound core configuration with the lowest energy of a given mass, size, and angular momentum.

The number of candidates for the lowest energy core state can be significantly reduced using the following heuristic argument. The energy of an ensemble of rotating cores is minimized when the mass is concentrated into the largest core. This is so because, for sufficiently small short-range cutoff, the total energy of a rotating self-gravitating system is dominated by the negative gravitational self-energy of the cores. The absolute value of gravitational self-energy of a core grows with the core mass faster than linearly for all reasonable forms of short-range cutoff. Hence, the energy is minimized by a configuration which consists of the principal core of the maximum possible mass, while the remaining mass is distributed to carry the given angular momentum in the most energy-efficient way. For this reason, the particle system in simulations always evolves towards the asymmetric configuration consisting of the single core which carries all the angular momentum itself [12]. Without a constraint that the system has to occupy a finite volume, the lowest-energy state would have consisted of a core containing all but one particle, with that particle having an orbit radius defined by the angular momentum constraint. Likewise, with the spatial localization constraint, the most mass- and energy-efficient way to carry the given angular momentum is to put the smallest possible mass on a circular orbit of the maximum allowed radius. Being evident for a two-body system (see, for example, Ref. [15]), the energy efficiency of circular orbits can be seen from the following argument: If a circular orbit is perturbed by adding a radial component to the velocity, the angular momentum is unchanged while the energy increases. Below, we consider two possible configurations consisting of the principal core and the remaining mass on a single circular orbit: A central core with a ring of N orbiting cores and a binary, generally asymmetric. A symmetric binary is a limiting case of both families. Examples are abundant in the universe and have been observed in the MF analysis [5,7] as possible equilibrium or metastable configurations. Other mechanically stable rotating core configurations that do not belong to these two families apparently have less mass concentrated into the largest core, and thus have higher energy.

Using simple mechanics, we will derive that the choice of the lowest energy state depends on the range of a short-range (or high-density) regularization and the angular momentum. For an intermediate range of the small-distance regularization and small angular momentum, the binary state has the lowest ground-state energy, which confirms the results of Ref. [8]. In a limit of the vanishing range of the cutoff, or for the higher angular momentum, the core-ring state becomes the equilibrium one. The paper is organized as follows: After this introduction we define the model more formally. In Sec. III we compare the ground-state energies of two families of systems: A central core with a multicore ring, and an asymmetric binary. A discussion and conclusion section completes the paper.

II. DEFINITION OF THE MODEL

Let us now formally define the model. We search for the lowest-energy state of a system of $M \gg 1$ self-gravitating unit mass particles with fixed total angular momentum L localized within a sphere of radius R. To make the problem analytically tractable, we will limit our consideration to the case when the range of small-distance (high-density) regularization is short. Hence, the volume of all condensed objects in the ground state is considered to be negligible compared to the volume of the system. In addition, to make a clear distinction from the nonrotating case, we focus our attention on the sufficiently high values of angular momentum L to exclude the configuration with the single spinning central core. Most of the analysis below is for a system of classical particles interacting via the attractive soft Coulomb potential $-(r^2+r_0^2)^{-1/2}$ (the gravitational constant is set to be unity). This simple form of short-range regularization is qualitatively equivalent to other forms of "softening," such as truncation of the Fourier or spherical harmonic expansions. For an interparticle distance r smaller than the respective softening radius (given by r_0, or by a wavelength of the highest untruncated harmonic function), all soft potentials tend to a harmonic oscillator potential. Consequently, for all soft potentials a condensed core with no particle motion (at zero temperature) is a pointlike object. This is different from, for example, the ground state of a system of fermions or hard core particles with the finite core volume. Qualitative arguments will be given to show that conclusions made for the system with soft potential also hold for systems with other forms of short-range regularization.

In addition to the notations r, E, and L for distance, energy, and absolute value of angular momentum, in the following we will also use the rescaled (universal) units for distance x, energy ϵ, and angular momentum λ (see, for example, Refs. [1,3,5])

$$x \equiv \frac{r}{R},$$

$$\epsilon \equiv \frac{ER}{M^2},$$

$$\lambda \equiv \frac{L}{\sqrt{M^3 R}}. \tag{1}$$

III. CORE-RING AND ASYMMETRIC BINARIES

First consider a ring, i.e., a system of N identical cores (point masses in the case of a soft potential) of mass M/N each moving on a circular orbit of radius $r \leqslant R$. Assume that the softening radius r_0 is much smaller than the distance between the neighboring cores, and the interaction between cores can be considered a bare gravitational one.

Mechanical equilibrium requires the centripetal acceleration of each core to be equal to the total force exerted on it by the others

$$\omega^2 r = \sum_{i=1}^{N-1} \frac{M \cos[(\pi - \phi_i)/2]}{2Nr^2(1-\cos\phi_i)}, \tag{2}$$

with $\phi_i = 2\pi i/N$. This gives the angular momentum of the system

$$L^2 = M^3 r \frac{1}{4N} \sum_{i=1}^{N-1} \frac{1}{\sin(\pi i/N)} \underset{N\to\infty}{\longrightarrow} M^3 r \frac{\ln N + C}{2\pi}, \tag{3}$$

where for large N the sum is replaced by an integral. This asymptotic expression turns out to be quite robust (less than 4% off, even for $N=2$), and the small numerical value of the constant $C \approx 0.126$ makes C negligible even for few-core systems.

It follows from (2) and (3) that a binary system ($N=2$) localized within a sphere of radius R cannot have rescaled angular momentum higher than

$$\lambda_{bin} = 1/\sqrt{8} \approx 0.35. \tag{4}$$

In a ring, the number of ring cores grows as $N \sim \exp(2\pi\lambda^2)$ for large λ. The increase of N, however, cannot continue indefinitely: For the interaction between cores to be similar to the bare gravitational potential $-1/r$, the distance between cores should be larger than the softening radius r_0. Hence, when the number of cores exceeds the corresponding limit $N > N_{max} \approx 2\pi r/r_0$, the cores may still get closer to each other, while the maximum force acting on each core saturates at the corresponding to N_{max} value. Consequently, the maximum angular momentum of a self-gravitating system with radius not greater than R is

$$\lambda_{max} \approx \sqrt{\frac{\ln(R/r_0)}{2\pi}} = \sqrt{\frac{|\ln x_0|}{2\pi}}. \tag{5}$$

Qualitatively similar estimate for the maximum angular momentum λ_{max} exists for systems with finite-size cores, such as those formed by ensembles of fermions or hard-core particles. For such systems, the maximum number of cores N_{max} corresponds to a merging of cores into a continuous (finite-volume) ring or torus. The radius of the body of the torus r_c (which is of the order of the radii of cores before merging), serves the role of a cutoff parameter r_0 in (5). This conclusion can also be reached using the following argument: One can split a continuous ring into N_{max} segments of size $\sim r_c$ and consider the interaction between them in the multipole expansion. The monopole-monopole interaction gives rise to the leading logarithmic term in (3) and (5), while the higher-order terms produce only $\mathcal{O}(r_c/R)^0$ corrections.

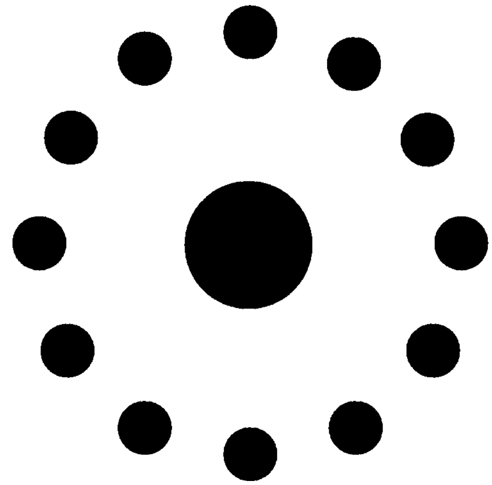

FIG. 1. A sketch of a central core-ring system.

Let us now consider a core-ring system consisting of a single central core of mass $M(1-\alpha)$ and a ring of mass $M\alpha$ consisting of $N \geqslant 2$ cores. This structure, sketched in Fig. 1, resembles the planet Saturn with its ring.

Similarly to Eq. (3), the equation of motion for an orbiting core yields, for the angular momentum of the system

$$L^2 = M^3 r \alpha^2 \left[\alpha \frac{\ln N}{2\pi} + (1-\alpha) \right]. \tag{6}$$

The total energy of the core-ring system consists of the gravitational self-energies of the central core and orbiting cores, and the energy of macroscopic rotation

$$\epsilon = -\frac{(1-\alpha)^2}{2x_0} - \frac{\alpha^2}{2Nx_0} - \frac{\alpha^3\{\alpha[\ln N/(2\pi) - 1] + 1\}^2}{2\lambda^2}. \tag{7}$$

To find the ground-state energy, this expression must be minimized with respect to α and N, $2 \leqslant N \leqslant 1/x_0$, taking into account the $r \leqslant R$ constraint, which has the form

$$\alpha^2 \left(1 - \alpha + \alpha \frac{\ln N}{2\pi} \right) \geqslant \lambda^2. \tag{8}$$

For sufficiently small x_0, the first term in (7) is the dominant one. Hence, the minimum energy is reached when N is increased to its saturation value $N \sim 1/x_0$ to minimize the relative ring mass α within the range allowed by (8). This is illustrated by the example presented in Fig. 2. The ground-state energy of the core-ring configuration with the angular momentum $\lambda \leqslant \ln N/2\pi$ is

$$\epsilon_{c\text{-}r} = -\frac{(1-\alpha)^2}{2x_0} - \frac{\alpha^2}{2} - \frac{\alpha}{2}\left(\frac{\alpha|\ln x_0|}{2\pi} + 1 - \alpha \right), \tag{9}$$

where $\alpha(\lambda)$ is the minimal ring mass allowed by (8) and $N = 1/x_0$.

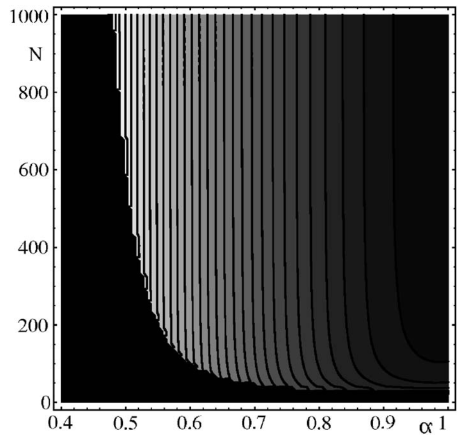

FIG. 2. Contour plot of ϵ vs α and N, defined by the Eq. (7) and constraint (8) for $\lambda=0.5$ and $x_0=10^{-3}$. The energy decreases (increases by absolute value) from dark to light, and reaches its minimum at the largest possible N, $N=1/x_0=1000$ and the smallest a allowed by (8), $\alpha\approx 0.49$. The black area corresponds to values of α and N which do not satisfy the localization constraint $r\leqslant R$ (8).

The mass distribution of the ground states of the core-ring systems with other forms of cutoff is similar. The self-energy of a spherical core of m particles can be expressed as $E_{\rm self}^{HC}=-C_{HC}m^{5/3}$ and $E_{\rm self}^{F}=-C_F m^{7/3}$ for systems of hard-core particles and fermions, respectively (see Ref. [16] for the energy of self-gravitating fermion ball). The constants C_{HC} and C_F, which depend on the hard-core radius and the number of internal degrees of freedom, play the role of $1/r_0$ in (9). For sufficiently large values of these constants (or equivalently, sufficiently small cores), the self-energy of the central core dominates over the self-energies of the ring cores and the energy of macroscopic motion. Hence, for a reasonable form of the short-range cutoff, the lowest energy core-ring configuration consists of the central core of the largest possible mass allowed by the localization constraint, and the orbiting ring of the system radius R which carries all the angular momentum.

Now let us consider a generally asymmetric binary [8] as another candidate for the lowest energy configuration with a moderate angular momentum $\lambda<\lambda_{bin}$ (4). Introducing an asymmetry parameter $0<\delta<1/2$ and the core separation $r\leqslant 2R$, denote the core masses and orbit radii as $M_1=\delta M$, $M_2=(1-\delta)M$, $r_1=(1-\delta)r$, and $r_2=\delta r$, correspondingly. The total energy consists of the energy of rotation and the gravitational self-energy of the cores, and reads in rescaled units

$$\epsilon=-\frac{[\delta(1-\delta)]^3}{2\lambda^2}-\frac{(1-\delta)^2+\delta^2}{2x_0}. \quad (10)$$

To find the ground-state energy, this expression has to be minimized with respect to δ while taking into account the constraint that the radius of the largest orbit r_1 should not exceed R

$$\delta^2(1-\delta)\geqslant\lambda^2. \quad (11)$$

For all reasonable values of x_0 ($x_0<32\lambda^2/3$), the gravitational self-energies of the cores dominates the total energy ϵ in Eq. (10). Hence, the energy is minimized

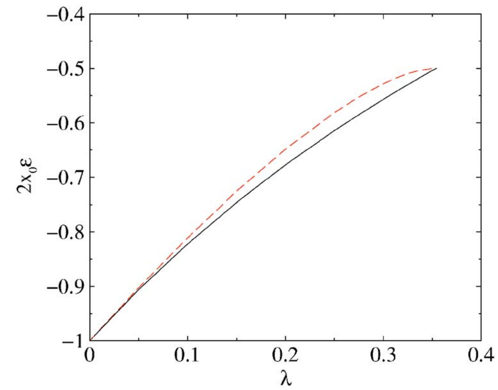

FIG. 3. (Color online) Plot of the rescaled ground-state energies of core-ring ϵ_{c-r} (solid line) and asymmetric binary e_b (dashed line) configurations vs angular momentum λ for $x_0=10^{-7}$.

$$\epsilon_{bin}=-\frac{\delta(1-\delta)^2}{2}-\frac{(1-\delta)^2+\delta^2}{2x_0}, \quad (12)$$

with the highest possible asymmetry, or the smallest possible value of $\delta\leqslant 1/2$ allowed by (11). The same is true for other forms of short-range regularization: Since the self-energy of a self-gravitating core grows faster than linearly with the core mass, a binary with the highest possible asymmetry will always have the lowest energy.

This confirms the conclusion made in Ref. [8] that for low energy an asymmetric binary has a higher entropy than a symmetric one, albeit with the restriction that orbits in a binary are gravitationally supported only for $\lambda<\lambda_{bin}$.

To compare the ground-state energies of core-halo and asymmetric binary configurations, the cubic equations (8) and (11) have to be solved for a given angular momentum $\lambda\leqslant\lambda_{bin}$, and the resulting α and δ are to be substituted into the expressions for energy (9) and (12). For sufficiently small cutoff radius $x_0\ll 1$, the self-energies of the cores, described by the first terms in (9) and (12), give dominant contributions to the total energy. Qualitatively, when the cutoff radius is still not too small, so that $|\ln x_0|/2\pi\ll 1$, the logarithmic term in (8) is negligible and $\alpha\approx\delta$. In this case, due to the noticeable contribution of the self-energy of the smaller core, $-\delta^2/2x_0$, the binary system has the lowest energy. On the other hand, if $|\ln x_0|/2\pi\gg 1$, the logarithmic term dominates in (8) and $\alpha<\delta$. In this case, the self-energy of the bigger central core of the core-ring system becomes smaller than the self-energies of both binary cores, and the core-ring system is the equilibrium one. The λ dependence of the binary and core-halo energies in the borderline case of $x_0\approx 10^{-7}$ is illustrated in Fig. 3. Hence, for $x_0<10^{-7}$, the core-ring configuration is the equilibrium one for all values of angular momentum λ, while for $x_0>10^{-7}$ there exists a range of $\lambda\leqslant\lambda_{bin}$ for which the binary system is the equilibrium one. Naturally, the energies of the core-ring and binary configuration coincide for $\lambda=0$ where both configurations are reduced to just one central core.

IV. DISCUSSION AND CONCLUSION

In the previous sections we considered the low-energy equilibrium states of rotating self-gravitating systems and arrived at the following conclusions.

To set a distinction between the physically relevant and irrelevant states, we suggested that the core part of the relevant states has to be moving on gravitationally supported orbits and not come in contact with the system boundary. In the low-energy or zero-temperature limit, such equilibrium states become localized core-only states and can exist without a container.

We found that, depending on the range of the short-distance regularization and the angular momentum, two possibilities exist for the equilibrium configurations of rotating self-gravitating systems: For an intermediate range of short-distance cutoff and small angular momentum $L \leq \sqrt{M^3 R/8}$, an asymmetric two-core binary configuration has the lowest ground-state energy and thus is the equilibrium one [8]. For $L > \sqrt{M^3 R/8}$, or for a very small range of the cutoff and arbitrary angular momentum, the equilibrium configuration consists of a central core and an equatorial ring.

The precise value of the cutoff range at which these two configurations have the same ground-state energy depends on the nature of the cutoff. For the soft-core interaction potential, the crossover between two ground states takes place when the softening radius r_0 satisfies $R/r_0 \approx 10^7$. For other forms of regularization, the role of r_0 is played by the radius of the body of the ring, or the radii of multiple "cores" that constitute the ring. The maximum angular momentum of a localized self-gravitating system scales as

$$L_{max} \underset{r_0/R \to 0}{\to} \sqrt{\frac{M^3 R \ln(R/r_0)}{2\pi}}.$$

In this limit the central core vanishes and the system consists only of a ring.

It would be desirable to check these conclusions by either particle simulations or the mean field analysis. Unfortunately, at this stage, none of these seems feasible. As shown in Ref. [12] and discussed in the Introduction, any finite-temperature particle simulations will lead to the physically irrelevant configuration with the single core sliding along the container wall. Another obstacle lies in the size of such a computation: To be able see the crossover between the binary and core-ring system one needs a ring consisting of $\approx 10^7$ cores, each consisting of at least one particle. A similar requirement on spatial resolution makes the mean field calculations very challenging too. Indeed, even the existing calculations with fairly large cutoff [5–8] may not have enough mesh points to resolve the core structure [17]. However, the predicted crossover ratio $R/r_0 \approx 10^7$ is not at all astrophysically irrelevant and may be encountered in the planetary disks and even Saturn rings. Given that the typical radii of rings around the big planets of the solar system is of the order of 10^5 km [18], the dominant presence of ring particles of the size of 1 cm and less will make the core-ring configuration thermodynamically more stable than the binary one.

The present consideration of the core-ring structure is rather schematic and at the current level does not allow us to explain the fine structure of rings as gaps and spikes. Neither is the Roche limit, which may set another bound on the existence of the low-orbit self-gravitating binaries due to the tidal interaction, considered. Yet, even the present level of modeling of core-ring structures permits one to make important thermodynamical predictions, and a distinction between the existing core-ring models. For example, the central core-ring ground-state configuration considered here may look similar to the core-ring configuration found for the system of fermions with fixed angular velocity in Ref. [11]. However, this similarity is only superficial: The ring observed in Ref. [11] is formed by particles which could not be supported gravitationally at the equator of the central core and were shedded off to the container wall, while in our case the ring is supported only gravitationally.

Finally, it is interesting to speculate on up to what energy or temperature does the correspondence between the equilibrium and lowest ground-state energy configurations hold? Indeed, as the energy increases and the mass of the halo becomes comparable or exceeds the combined mass of condensed objects (cores and rings), most of the angular momentum is carried by the halo. This may lead to a reduction and complete disappearance of a ring. This scenario looks especially plausible if the angular momentum is noticeably smaller than L_{max}, so a gravitationally supported structure with smaller moment of inertia than that of an equatorial ring can carry it. In addition to numerical methods, this scenario can be analyzed analytically by approximating a halo as a uniform gas, as done by Chavanis [19] in the case of a nonrotating state. We leave this analysis for future research.

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